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A model with coexistence of two kinds of Bose condensation

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Abstract. We present an exactly soluble boson model which manifests two kinds of condensation. They occur in two stages: for intermediate densities one has a *non-conventional* Bose condensation in the lowest mode $k = 0$, which is due to a diagonal perturbation of the imperfect Bose gas Hamiltonian, whereas for large densities ρ, this condensation coexists with *conventional (generalized, non-extensive)* Bose–Einstein condensation in non-zero modes condensation, corresponding to the standard mechanism of saturation.

1. Introduction

Since its first description by Einstein [1] in 1925, it has been known that *conventional* Bose–Einstein condensation with macroscopic occupation of a single level is a very subtle matter. For example, its magnitude strongly depends on the shape of container and on the way one takes the thermodynamic limit, see e.g. [2, 3] and appendix A. It was Casimir [4] who showed that in a long prism it is possible for condensation in the perfect Bose gas (PBG) to occur in a 'narrow band' rather than in a single level. This was an example of *generalized* Bose–Einstein condensation, a concept introduced earlier by Girardeau [5]. The first rigorous treatment of this observation for the PBG was due to van den Berg, Lewis and Pulè [3, 6–10]. They proposed a classification of generalized condensation types; according to this classification the condensation in a single level is of type I, see appendix A.

The salient feature of conventional Bose–Einstein condensation (generalized or not) is that it appears in non-interacting systems of bosons (this does not rule out external potentials) as soon as the total particle density becomes larger than some critical value. Therefore, behind this kind of condensation there exists a *saturation mechanism* related to the Bose statistics of particles. It was demonstrated in [11] that exactly the same mechanism is responsible for Bose–Einstein condensation in a system of bosons with repulsive interaction, commonly called the imperfect Bose gas (IBG). In a recent paper [12] it was shown that, instead of the geometry of the container, a judicious choice of repulsive interaction may split initial single-level condensation (type I) into non-extensive (type III) condensation, when no levels are macroscopically occupied.

Therefore, the concept of (generalized) conventional Bose–Einstein condensation caused by the mechanism of saturation fits well for bosons with repulsive interaction.

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Since bosons are very sensitive to attraction, there is another kind of condensation (*non-conventional* Bose condensation) induced by this interaction [13–15]. Again, this kind of condensation appears when the total particle density (or chemical potential) becomes larger than some critical value, but this is an attractive interaction (and not simply Bose statistics) which defines the magnitude of the condensate and its behaviour. To escape the collapse, an attractive interaction in a boson system should be stabilized by a repulsion. Therefore, conventional and non-conventional condensation may coexist.

The aim of this paper is to study a model which has both kinds of condensation. The non-conventional one is due to an attraction term in the Hamiltonian of the model. This condensation starts at the single lowest level for moderate densities (negative chemical potentials) and saturates at some critical density. It is above this threshold that the *conventional* Bose–Einstein condensation appears to absorb the increasing total particle density (the saturation mechanism).

At these densities both kinds of condensation coexist. Moreover, the repulsive interaction in our model is such that Bose–Einstein condensation splits into the non-extensive one, i.e., into the generalized type III condensation. Since it is known that Bose systems manifesting condensation are far from perfect, we hope that our model will give more insight into possible scenarios for condensation in real systems. For example, in a condensate of sodium atoms, interaction seems to predominate compared to kinetic energy [16]. Therefore, condensation in trapped alkali dilute gases [16–18] should be a combination of *non-conventional* and *conventional* Bose condensation.

To fix the notation we first recall the IBG model introduced by Huang [19]. It is a system of identical bosons of mass m enclosed in a cube $\Lambda \subset \mathbb{R}^d$, of volume $V = |\Lambda|$, centred at the origin defined by the Hamiltonian

$$H_\Lambda^{\mathrm{IBG}} = \sum_{k\in\Lambda^*} \varepsilon_k a_k^* a_k + \frac{\lambda}{V} N_\Lambda^2 \qquad \varepsilon_k \equiv \hbar^2 k^2/2m \quad \lambda > 0 \tag{1.1}$$

where $N_\Lambda = \sum_{k\in\Lambda^*} a_k^* a_k \equiv \sum_{k\in\Lambda^*} N_k$ is the particle-number operator and ε_k corresponds to the one-particle kinetic energy. Here $\{a_k^\#\}_{k\in\Lambda^*}$ are the boson creation/annihilation operators in the boson Fock space $\mathcal{F}_\Lambda$ over $L^2(\Lambda)$, corresponding to the second quantization in the box $\Lambda = \times_{\alpha=1}^d L$ with periodic boundary conditions, i.e. to the dual

$$\Lambda^* = \left\{ k \in \mathbb{R}^d : k_\alpha = \frac{2\pi n_\alpha}{L}, n_\alpha = 0, \pm1, \pm2, \ldots; \alpha = 1, 2, \ldots, d \right\}.$$

Then, for $d > 2$, at a given temperature $\theta = \beta^{-1}$ and a total particle density $\rho > \rho_c^P(\theta)$ (here $\rho_c^P(\theta) \equiv \rho^P(\theta^{-1}, \mu = 0)$, where $\rho^P(\beta, \mu)$ is the particle density of the PBG in the grand-canonical ensemble) the IBG manifests a *conventional* Bose–Einstein condensation of type I [11, 20, 21], i.e. a macroscopic occupation of only the single-particle ground-state level $k = 0$. See [7, 8] or appendix A for a classification of *conventional* Bose–Einstein condensations.

However, in a recent paper [12] it was shown that the IBG (1.1) perturbed by the repulsive diagonal interaction

$$\widetilde{U}_\Lambda = \frac{\lambda}{2V} \sum_{k\in\Lambda^*} N_k^2 \qquad \lambda > 0 \tag{1.2}$$

demonstrates the Bose–Einstein condensation which occurs again for densities $\rho > \rho_c^P(\theta)$ (or $\mu > 2\lambda\rho_c^P(\theta) \equiv \mu_c^I(\theta)$), but now it spreads into the Bose–Einstein condensation of type III. This is a *non-extensive* condensation, when *no* single-particle levels are macroscopically occupied (see appendix A). This model for $\lambda > 0$ was introduced in [22]. In what follows we call it the Michoel–Schröder–Verbeure (MSV) model:

$$H_\Lambda^{\mathrm{MSV}} \equiv H_\Lambda^{\mathrm{IBG}} + \widetilde{U}_\Lambda. \tag{1.3}$$

Then the conventional Bose–Einstein condensation of type III means that

$$\lim_{\Lambda} \frac{\langle N_k \rangle_{H_\Lambda^{\mathrm{MSV}}}}{V} = 0 \qquad k \in \Lambda^*$$

for any ρ, whereas the double limit

$$\lim_{\delta\to 0^+} \lim_{\Lambda} \frac{1}{V} \sum_{\{k\in\Lambda^*, 0\leqslant \|k\|\leqslant\delta\}} \langle N_k \rangle_{H_\Lambda^{\mathrm{MSV}}} = \rho - \rho_c^{\mathrm{P}}(\theta) > 0$$

for $\mu > \mu_c^I(\theta)$. Here we denote by $\langle - \rangle_{H_\Lambda^{\mathrm{MSV}}}(\beta, \mu)$, $\beta \geqslant 0$, $\mu \in \mathbb{R}^1$, the grand-canonical Gibbs state for the Hamiltonian H_Λ^{MSV}. Note that the model $H_\Lambda^{\mathrm{IBG}} - \widetilde{U}_\Lambda$ is rigorously studied in [23]. There it was shown that Bose–Einstein condensation is of type I only in the zero-mode $k = 0$.

The fact that a gentle repulsive interaction may produce a generalized *non-extensive* Bose–Einstein condensation without any change of corresponding pressure has been also shown in our recent paper [13]. This was done in context of a system:

$$H_\Lambda^0 = \sum_{k\in\Lambda^*\setminus\{0\}} \varepsilon_k a_k^* a_k + \varepsilon_0 a_0^* a_0 + \frac{g_0}{2V} a_0^* a_0^* a_0 a_0 \tag{1.4}$$

with $\varepsilon_0 (\neq \varepsilon_{k=0}) \in \mathbb{R}^1$ and $g_0 > 0$, perturbed by the interaction

$$U_\Lambda = \frac{1}{V} \sum_{k\in\Lambda^*\setminus\{0\}} g_k(V) a_k^* a_k^* a_k a_k \qquad 0 < g_- \leqslant g_k(V) \leqslant \gamma_k V^{\alpha_k} \tag{1.5}$$

with $\alpha_k \leqslant \alpha_+ < 1$ and $0 < \gamma_k \leqslant \gamma_+$. This perturbation U_Λ (similar to the interaction $\widetilde{U}_\Lambda$ when $g_k = \lambda$) leads to the Hamiltonian [13]

$$H_\Lambda^{\mathrm{BZ}} \equiv H_\Lambda^0 + U_\Lambda. \tag{1.6}$$

In contrast to the MSV model, the grand-canonical pressure for our model (1.6)

$$p_\Lambda^{\mathrm{BZ}}(\beta, \mu) \equiv p_\Lambda[H_\Lambda^{\mathrm{BZ}}] \equiv \frac{1}{\beta V} \ln \mathrm{Tr}_{\mathcal{F}_\Lambda} \mathrm{e}^{-\beta(H_\Lambda^{\mathrm{BZ}} - \mu N_\Lambda)} \tag{1.7}$$

in the thermodynamic limit is only defined in the domain $Q = \{\mu \leqslant 0\} \times \{\theta \geqslant 0\}$. Here μ is the chemical potential of the grand-canonical ensemble. In the thermodynamic limit this pressure is equal to

$$p^{\mathrm{BZ}}(\beta, \mu) \equiv \lim_{\Lambda} p_\Lambda^{\mathrm{BZ}}(\beta, \mu) = \lim_{\Lambda} p_\Lambda[H_\Lambda^0] = p^{\mathrm{P}}(\beta, \mu) - \inf_{\rho_0\geqslant 0} \left[(\varepsilon_0 - \mu)\rho_0 + \frac{g_0 \rho_0^2}{2} \right] \tag{1.8}$$

see [13]. Here $p^{\mathrm{P}}(\beta, \mu)$ is the pressure of the PBG in the thermodynamic limit. Notice that the pressure (1.8) is independent of the parameters $\{g_k(V)\}_{k\in\Lambda^*\setminus\{0\}}$, i.e. of interaction (1.5).

Remark 1.1. Let domain D_{ε_0} be defined by

$$D_{\varepsilon_0} \equiv \{(\theta, \mu) \in Q : p^{\mathrm{P}}(\beta, \mu) < p^{\mathrm{BZ}}(\beta, \mu)\}. \tag{1.9}$$

Then the thermodynamic limit (1.8) states that to ensure $D_{\varepsilon_0} \neq \{\emptyset\}$, the parameter ε_0 must be negative, i.e.

$$D_{\varepsilon_0} = \{(\theta, \mu) \in Q : \varepsilon_0 < \mu \leqslant 0\}. \tag{1.10}$$

Below we only consider the case $\varepsilon_0 < 0$ and $d > 2$.

We denote by $\rho_\Lambda^{\mathrm{BZ}}(\beta, \mu)$, the total particle density in the grand-canonical ensemble for the model H_Λ^{BZ}:

$$\rho_\Lambda^{\mathrm{BZ}}(\beta, \mu) \equiv \left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda^{\mathrm{BZ}}} (\beta, \mu). \tag{1.11}$$

Then $\rho^{\mathrm{BZ}}(\beta,\mu) \equiv \lim_\Lambda \rho_\Lambda^{\mathrm{BZ}}(\beta,\mu)$ is the corresponding thermodynamic limit which, according to [13], is equal to

$$\rho^{\mathrm{BZ}}(\beta,\mu) = \rho^{\mathrm{P}}(\beta,\mu) \tag{1.12}$$

for $(\theta, \mu \leqslant \varepsilon_0)$, and to

$$\rho^{\mathrm{BZ}}(\beta,\mu) = \rho^{\mathrm{P}}(\beta,\mu) + \frac{\mu-\varepsilon_0}{g_0} \tag{1.13}$$

for $(\theta, \varepsilon_0 < \mu < 0)$. Note that for $d > 2$ there is a finite critical density

$$\rho_c^{\mathrm{BZ}}(\theta) \equiv \sup_{\mu\leqslant 0} \rho^{\mathrm{BZ}}(\theta^{-1},\mu) = \rho^{\mathrm{BZ}}(\theta^{-1},\mu=0) = \rho_c^{\mathrm{P}}(\theta) - \frac{\varepsilon_0}{g_0} < +\infty \tag{1.14}$$

in this model.

Proposition 1.2. *[13] Let* $\rho > \rho_c^{\mathrm{BZ}}(\theta)$ $(d>2)$ *and* $0 < g_- \leqslant g_k(V) \leqslant \gamma_k V^{\alpha_k}$ *for* $k \in \Lambda^*\backslash\{0\}$, *with* $\alpha_k \leqslant \alpha_+ < 1$ *and* $0 < \gamma_k \leqslant \gamma_+$. *Then for any* $\varepsilon_0 < 0$ *we have:*

(i) a condensation in the mode $k=0$ *(even if* $d<3$*), i.e.*

$$\rho_0^{\mathrm{BZ}}(\theta,\mu) \equiv \lim_\Lambda \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_\Lambda^{\mathrm{BZ}}} = \left\{ \begin{array}{ll} 0, & \textit{for } (\theta,\mu) \in Q \setminus D_{\varepsilon_0} \\ \left(\frac{\mu-\varepsilon_0}{g_0}\right), & \textit{for } (\theta,\mu) \in D_{\varepsilon_0} \end{array} \right\} \tag{1.15}$$

(ii) for any $\varepsilon_0 \in \mathbb{R}^1$

$$\lim_\Lambda \left\langle \frac{a_k^* a_k}{V} \right\rangle_{H_\Lambda^{\mathrm{BZ}}} = 0 \qquad k \in \Lambda^* \setminus \{0\} \tag{1.16}$$

i.e. there is no macroscopic occupation of modes $k \neq 0$ *but we have a generalized (non-extensive) Bose–Einstein condensation:*

$$\lim_{\delta\to 0^+} \lim_\Lambda \frac{1}{V} \sum_{\{k\in\Lambda^*:0<\|k\|<\delta\}} \langle N_k \rangle_{H_\Lambda^{\mathrm{BZ}}} = \rho - \rho_c^{\mathrm{BZ}}(\theta) > 0 \tag{1.17}$$

which for $\varepsilon_0 < 0$ *coexists with the condensation in the mode* $k = 0$ *if* $(\theta,\mu) \in D_{\varepsilon_0}$.

Therefore, proposition 1.2 implies the *coexistence* of *two kinds* of Bose condensation in model (1.6) for densities $\rho > \rho_c^{\mathrm{BZ}}(\theta)$:

- a Bose condensation in the single mode $k = 0$ due to the term $(\varepsilon_0 a_0^* a_0)$ which for $\varepsilon_0 < 0$ mimics an *attraction* by an external potential [24] giving rise to a *non-conventional* condensation of type I (see appendix A for the classification);
- a *conventional* Bose–Einstein condensation due to *saturation* of the total particle density, where (similar to the MSV model) the type III condensation is due to the elastic repulsive interaction U_Λ (1.5) of bosons in modes $k \neq 0$. Therefore, interaction (1.5) is decisive for the formation of the non-extensive Bose–Einstein condensation in model (1.6), while it has no influence on the pressure (1.8).

The aim of this paper is to study a modification of model (1.6) which is, similar to (1.3), stabilized by the IBG interaction (1.1):

$$H_\Lambda = \sum_{k\in\Lambda^*\backslash\{0\}} \varepsilon_k a_k^* a_k + \varepsilon_0 a_0^* a_0 + \frac{g_0}{2V} N_0^2 + \frac{\lambda}{V} N_\Lambda^2 + \frac{g}{2V} \sum_{k\in\Lambda^*\backslash\{0\}} N_k^2. \tag{1.18}$$

Here $\lambda > 0$, $g_0 > 0$, $g > 0$, but $\varepsilon_0 < 0$. Note that the model H_Λ for $\varepsilon_0 = 0$ and $\lambda = g = g_0$ coincides with the MSV model (1.3).

In section 2 we show that for this generalization of models (1.3) and (1.6), the free-energy density or the pressure can be calculated exactly in the thermodynamic limit. Different kinds of Bose condensations, which occur in this model (1.18), are described in section 3. We reserve section 4 for concluding remarks and discussions. A classification of Bose condensations and some technical statements are presented in the appendices.

2. Thermodynamic study

First we consider our model (1.18) in the *canonical ensemble* (β, ρ). This essentially simplifies the thermodynamic study of the model. Let $f_\Lambda(\beta, \rho = \frac{n}{V})$ be the corresponding free-energy density, i.e.

$$f_\Lambda(\beta, \rho) \equiv -\frac{1}{\beta V} \ln \mathrm{Tr}_{\mathcal{H}^n_{\Lambda,S}}(\mathrm{e}^{-\beta H_\Lambda}) \tag{2.1}$$

where $\mathcal{H}^n_{\Lambda,S} \equiv S(\otimes^n_{i=1} L^2(\Lambda))$ is the symmetrized n-particle Hilbert space.

Theorem 2.1. *Let* $\lambda > 0$, $g > 0$, $g_0 > 0$ *and* $\varepsilon_0 < 0$*, then we get*

$$f(\beta, \rho) \equiv \lim_\Lambda f_\Lambda(\beta, \rho) = \lambda\rho^2 + \inf_{\rho_0 \in [0,\rho]} \left\{ \varepsilon_0\rho_0 + \frac{g_0}{2}\rho_0^2 + f^{\mathrm{P}}(\beta, \rho - \rho_0) \right\} \tag{2.2}$$

i.e. the limit is independent of g*, provided* $g > 0$*. Here* $f^{\mathrm{P}}(\beta, \rho)$ *is the free energy of the PBG in the thermodynamic limit, i.e.*

$$f^{\mathrm{P}}(\beta, \rho) \equiv \lim_\Lambda f^{\mathrm{P}}_\Lambda(\beta, \rho) \tag{2.3}$$

with

$$f^{\mathrm{P}}_\Lambda(\beta, \rho) \equiv -\frac{1}{\beta V} \ln \sum_{\{n_k = 0,1,2,\ldots\}_{k \in \Lambda^*}} \mathrm{e}^{-\beta(\sum_{k\in\Lambda^*} \varepsilon_k n_k)} \delta_{\sum_{k\in\Lambda^*} n_k = [\rho V]} \tag{2.4}$$

and $[x]$ *denotes the integer part of* $x \geqslant 0$.

Proof. By (1.18) and (2.1) we get

$$f_\Lambda(\beta, \rho) = -\frac{1}{\beta V} \ln \left\{ \sum_{n_0=0}^{[\rho V]} \mathrm{e}^{-\beta V h(\rho, \frac{n_0}{V})} \right\} + \lambda\rho^2 \tag{2.5}$$

where

$$h_\Lambda(\rho, \rho_0) \equiv \varepsilon_0\rho_0 + \frac{g_0}{2}\rho_0^2 - \frac{1}{\beta V} \ln \sum_{\{n_k=0,1,2,\ldots\}_{k\in\Lambda^*\setminus\{0\}}} \mathrm{e}^{-\beta(\sum_{k\in\Lambda^*\setminus\{0\}}[\varepsilon_k n_k + \frac{g}{2V} n_k^2])} \delta_{\sum_{k\neq 0} n_k = [\rho V] - [\rho_0 V]}. \tag{2.6}$$

By (2.5) one obtains the estimate

$$\lambda\rho^2 + \inf_{\rho_0\in[0,\rho]} h_\Lambda(\rho, \rho_0) - \frac{1}{\beta V} \ln([\rho V] + 1) \leqslant f_\Lambda(\beta, \rho) \leqslant \lambda\rho^2 + \inf_{\rho_0\in[0,\rho]} h_\Lambda(\rho, \rho_0)$$

which in the thermodynamic limit gives

$$f(\beta, \rho) \equiv \lim_\Lambda f_\Lambda(\beta, \rho) = \lambda\rho^2 + \lim_\Lambda \inf_{\rho_0\in[0,\rho]} h_\Lambda(\rho, \rho_0). \tag{2.7}$$

Note that (2.6) can be rewritten as

$$h_\Lambda(\rho, \rho_0) = \varepsilon_0\rho_0 + \frac{g_0}{2}\rho_0^2 - \frac{1}{\beta V} \ln\langle \mathrm{e}^{-\frac{\beta g}{2V}\sum_{k\in\Lambda^*\setminus\{0\}} n_k^2} \rangle_{\widetilde{H}^{\mathrm{P}}_\Lambda}(\beta, \rho - \rho_0) \tag{2.8}$$

where $\langle - \rangle_{\widetilde{H}^{\mathrm{P}}_\Lambda}(\beta, \rho - \rho_0)$ is the canonical Gibbs state for the PBG with *excluded* mode $k = 0$ for density $\rho - \rho_0$, with the corresponding free-energy density $\widetilde{f}^{\mathrm{P}}_\Lambda(\beta, \rho)$ defined by (2.4) for $k \in \Lambda^* \setminus \{0\}$. Since

$$\lim_\Lambda \widetilde{f}^{\mathrm{P}}_\Lambda(\beta, \rho) = \lim_\Lambda f^{\mathrm{P}}_\Lambda(\beta, \rho)$$

the Jensen inequality

$$\langle \mathrm{e}^{-\frac{\beta g}{2V}\sum_{k\in\Lambda^*\setminus\{0\}} n_k^2}\rangle_{\widetilde{H}_\Lambda^{\mathrm{P}}} \geqslant \mathrm{e}^{-\frac{\beta g}{2V}\langle \sum_{k\in\Lambda^*\setminus\{0\}} n_k^2\rangle_{\widetilde{H}_\Lambda^{\mathrm{P}}}}$$

and (2.8) imply the estimate

$$\lim_\Lambda h_\Lambda(\rho,\rho_0) \leqslant \varepsilon_0\rho_0 + \frac{g_0}{2}\rho_0^2 + f^{\mathrm{P}}(\beta,\rho-\rho_0). \tag{2.9}$$

Moreover, since

$$\mathrm{e}^{-\frac{\beta g}{2V}n_k^2} \leqslant 1$$

by (2.6) we have

$$h_\Lambda(\rho,\rho_0) \geqslant \varepsilon_0\rho_0 + \frac{g_0}{2}\rho_0^2 + f_\Lambda^{\mathrm{P}}(\beta,\rho-\rho_0)$$

which together with (2.9) gives (2.2). □

Remark 2.2. Let us denote by $f_\Lambda^{\mathrm{BZ}}(\beta,\rho)$ the free-energy density corresponding to H_Λ^{BZ} (1.6) with $g_k(V) = g/2$, i.e.

$$f_\Lambda^{\mathrm{BZ}}(\beta,\rho) \equiv -\frac{1}{\beta V}\ln \mathrm{Tr}_{\mathcal{H}_{\Lambda,S}^n}(\mathrm{e}^{-\beta H_\Lambda^{\mathrm{BZ}}}).$$

Then (1.6), (1.18) and (2.1) imply that

$$f_\Lambda(\beta,\rho) = \lambda\rho^2 + f_\Lambda^{\mathrm{BZ}}(\beta,\rho)$$

from which, by theorem 2.1, we deduce

$$f^{\mathrm{BZ}}(\beta,\rho) \equiv \lim_\Lambda f_\Lambda^{\mathrm{BZ}}(\beta,\rho) = \inf_{\rho_0\in[0,\rho]}\left\{\varepsilon_0\rho_0 + \frac{g_0}{2}\rho_0^2 + f^{\mathrm{P}}(\beta,\rho-\rho_0)\right\} \tag{2.10}$$

and

$$f(\beta,\rho) = \lambda\rho^2 + f^{\mathrm{BZ}}(\beta,\rho). \tag{2.11}$$

By explicit calculation, one checks the convexity of $f^{\mathrm{BZ}}(\beta,\rho)$ as a function of ρ. Therefore, the same is true for $f(\beta,\rho)$, see (2.2) and (2.11).

Remark 2.3. Since the pressure $p^{\mathrm{BZ}}(\beta,\mu)$ is a Legendre transform of the corresponding free-energy density $f^{\mathrm{BZ}}(\beta,\rho)$, we get from (2.10) that

$$\begin{aligned} p^{\mathrm{BZ}}(\beta,\mu) &= \sup_{\rho\geqslant 0}\{\mu\rho - f^{\mathrm{BZ}}(\beta,\rho)\} \\ &= \sup_{\rho_0\geqslant 0}\left\{\sup_{\rho\geqslant\rho_0}\left\{\mu\rho_0 - \varepsilon_0\rho_0 - \frac{g_0}{2}\rho_0^2 + \mu(\rho-\rho_0) - f^{\mathrm{P}}(\beta,\rho-\rho_0)\right\}\right\} \\ &= \sup_{\rho_0\geqslant 0}\left\{p^{\mathrm{P}}(\beta,\mu) - (\varepsilon_0-\mu)\rho_0 - \frac{g_0}{2}\rho_0^2\right\} \end{aligned}$$

which coincides with (1.8) found in [13].

Now we consider our model (1.18) in the *grand-canonical ensemble* (β,μ). Let

$$p_\Lambda(\beta,\mu) \equiv \frac{1}{\beta V}\ln \mathrm{Tr}_{\mathcal{F}_\Lambda}\,\mathrm{e}^{-\beta(H_\Lambda-\mu N_\Lambda)}$$

be the grand-canonical pressure corresponding to (1.18).

Theorem 2.4. *Let* $\lambda > 0$, $g_0 > 0$, $g > 0$, *and* $\varepsilon_0 < 0$, *then:*

(i) the domain of stability of H_Λ, i.e.

$$\widetilde{Q} \equiv \{(\theta \geqslant 0, \mu \in \mathbb{R}^1) : \lim_{\Lambda} p_\Lambda(\beta, \mu) < +\infty\} \tag{2.12}$$

is equal to $\widetilde{Q} = \{\theta \geqslant 0\} \times \{\mu \in \mathbb{R}^1\}$;

(ii) in the thermodynamic limit one gets

$$p(\beta, \mu) \equiv \lim_{\Lambda} p_\Lambda(\beta, \mu) = \inf_{\alpha \leqslant 0} \left\{ p^{\mathrm{BZ}}(\beta, \alpha) + \frac{(\mu - \alpha)^2}{4\lambda} \right\} \tag{2.13}$$

for $(\theta, \mu) \in \widetilde{Q}$, where $p^{\mathrm{BZ}}(\beta, \mu)$ is the pressure defined by (1.8). Therefore, the pressure (2.13) is independent of the parameter g provided it is positive.

Proof. (i) Note that the Hamiltonian H_Λ (1.18) is superstable, i.e. there are $B = -\varepsilon_0$ and $C = \lambda$ such that

$$H_\Lambda \geqslant -N_\Lambda B + \frac{C}{V} N_\Lambda^2 \tag{2.14}$$

for any box Λ. Therefore, by (2.14) we obtain that the infinite volume limit (2.13) exists for any $\mu \in \mathbb{R}^1$.

(ii) Since the pressure $p(\beta, \mu)$ is in fact a Legendre transform of the corresponding free-energy density $f(\beta, \rho)$ (2.2) or (2.11), by theorem 2.1 we get

$$p(\beta, \mu) = \sup_{\rho \geqslant 0} \{\mu\rho - f(\beta, \rho)\} = \sup_{\rho \geqslant 0} \{\mu\rho - \lambda\rho^2 - f^{\mathrm{BZ}}(\beta, \rho)\} \tag{2.15}$$

with $f^{\mathrm{BZ}}(\beta, \rho)$ defined by (2.10). Straightforward calculations give that

$$\inf_{\alpha \leqslant 0} \left\{ \alpha\rho + \frac{(\mu - \alpha)^2}{4\lambda} - f^{\mathrm{BZ}}(\beta, \rho) \right\} = \mu\rho - \lambda\rho^2 - f^{\mathrm{BZ}}(\beta, \rho)$$

and thus (2.15) takes the form

$$p(\beta, \mu) = \sup_{\rho \geqslant 0} \left\{ \inf_{\alpha \leqslant 0} \left\{ \alpha\rho + \frac{(\mu - \alpha)^2}{4\lambda} - f^{\mathrm{BZ}}(\beta, \rho) \right\} \right\}. \tag{2.16}$$

Note that the $\sup_{\rho \geqslant 0}$ and $\inf_{\alpha \leqslant 0}$ do not generally commute. However, convexity of the free-energy density $f^{\mathrm{BZ}}(\beta, \rho)$ (see remark 2.2) implies that

$$F(\rho, \alpha) \equiv \alpha\rho + \frac{(\mu - \alpha)^2}{4\lambda} - f^{\mathrm{BZ}}(\beta, \rho) \tag{2.17}$$

is a strictly concave function of ρ and a strictly convex function of α, see figure 1. This ensures the uniqueness of the stationary point $(\widetilde{\rho}, \widetilde{\alpha})$ corresponding to

$$\partial_\alpha F(\widetilde{\rho}, \widetilde{\alpha}) = 0$$
$$\partial_\rho F(\widetilde{\rho}, \widetilde{\alpha}) = 0.$$

Therefore,

$$F(\widetilde{\rho}, \widetilde{\alpha}) = \sup_{\rho \geqslant 0} \left\{ \inf_{\alpha \leqslant 0} \{F(\rho, \alpha)\} \right\} = \inf_{\alpha \leqslant 0} \left\{ \sup_{\rho \geqslant 0} \{F(\rho, \alpha)\} \right\}. \tag{2.18}$$

Since

$$\sup_{\rho \geqslant 0} F(\rho, \alpha) = \left\{ \frac{(\mu - \alpha)^2}{4\lambda} + p^{\mathrm{BZ}}(\beta, \alpha) \right\}$$

(2.16)–(2.18) imply (2.13). □

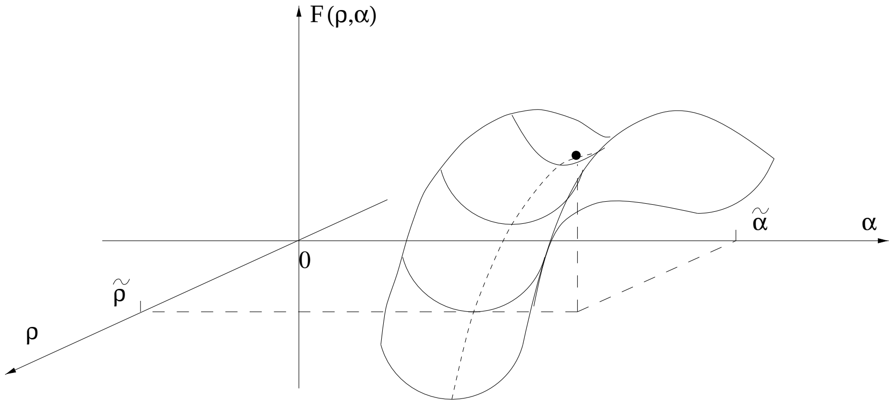

Figure 1. Illustration of the function $F(\rho, \alpha)$.

3. Bose condensations

Let $\rho_\Lambda(\beta, \mu)$ denote the grand-canonical total particle density corresponding to model (1.18), i.e.

$$\rho_\Lambda(\beta, \mu) \equiv \left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda} = \partial_\mu p_\Lambda(\beta, \mu) \tag{3.1}$$

where $\langle - \rangle_{H_\Lambda}(\beta, \mu)$ represents the grand-canonical Gibbs state for the Hamiltonian H_Λ (1.18).

Theorem 3.1. *For $(\theta, \mu) \in \widetilde{Q}$ (2.12) we have*

$$\rho(\beta, \mu) \equiv \lim_{\Lambda} \rho_\Lambda(\beta, \mu) = \rho^{\mathrm{BZ}}(\beta, \hat{\alpha}(\beta, \mu)). \tag{3.2}$$

Here $\widehat{\alpha}(\beta, \mu) \leqslant 0$ is a unique solution of the equation

$$\rho^{\mathrm{BZ}}(\beta, \alpha) + \frac{(\alpha - \mu)}{2\lambda} = 0 \tag{3.3}$$

when $\mu \leqslant \mu_c^{\mathrm{BZ}}(\theta) \equiv 2\lambda\rho_c^{\mathrm{BZ}}(\theta)$, whereas for $\mu > \mu_c^{\mathrm{BZ}}(\theta)$ one gets (see figure 2)

$$\rho(\beta, \mu) \equiv \lim_{\Lambda} \rho_\Lambda(\beta, \mu) = \frac{\mu}{2\lambda}. \tag{3.4}$$

Here $\rho_c^{\mathrm{BZ}}(\theta)$ is defined above by (1.14).

Proof. Let $\widetilde{\alpha}(\beta, \mu) \leqslant 0$ be defined by (2.13), i.e.

$$p(\beta, \mu) = \inf_{\alpha \leqslant 0} \left\{ p^{\mathrm{BZ}}(\beta, \alpha) + \frac{(\mu - \alpha)^2}{4\lambda} \right\} = p^{\mathrm{BZ}}(\beta, \widetilde{\alpha}(\beta, \mu)) + \frac{(\mu - \widetilde{\alpha}(\beta, \mu))^2}{4\lambda}. \tag{3.5}$$

Since

$$\partial_\alpha \left[p^{\mathrm{BZ}}(\beta, \alpha) + \frac{(\mu - \alpha)^2}{4\lambda} \right] = \rho^{\mathrm{BZ}}(\beta, \alpha) + \frac{(\alpha - \mu)}{2\lambda} \tag{3.6}$$

then for $\mu \leqslant \mu_c^{\mathrm{BZ}}(\theta) = 2\lambda\rho_c^{\mathrm{BZ}}(\theta)$ (see (1.14)) there exists a unique solution $\widehat{\alpha}(\beta, \mu) \leqslant 0$ of (3.3) which coincides with $\widetilde{\alpha}(\beta, \mu)$ in (3.5). Since $\{p_\Lambda(\beta, \mu)\}_\Lambda$ are convex functions of $\mu \in \mathbb{R}^1$, then combining (3.1) and (3.5) with the Griffiths lemma (see [25, 26] or appendix B) we obtain the thermodynamic limit for the total particle density

$$\rho(\beta, \mu) = \partial_\mu p(\beta, \mu) = \frac{(\mu - \widetilde{\alpha}(\beta, \mu))}{2\lambda}.$$

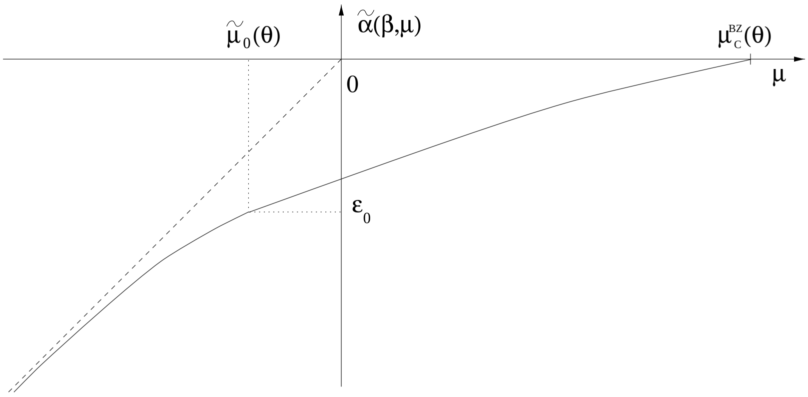

Figure 2. Illustration of the function $\widetilde{\alpha}(\beta, \mu)$ defined by equation (3.5).

This together with (3.3) gives (3.2).

Now let $\mu > \mu_c^{\mathrm{BZ}}(\theta)$. Then by definitions of $\mu_c^{\mathrm{BZ}}(\theta)$ and $\rho_c^{\mathrm{BZ}}(\theta)$ (see (1.14)) one gets

$$\partial_\alpha \left[p^{\mathrm{BZ}}(\beta, \alpha) + \frac{(\mu - \alpha)^2}{4\lambda} \right] = \rho^{\mathrm{BZ}}(\beta, \alpha) + \frac{(\alpha - \mu)}{2\lambda} \leqslant 0.$$

This implies that

$$p(\beta, \mu) = \inf_{\alpha \leqslant 0} \left\{ p^{\mathrm{BZ}}(\beta, \alpha) + \frac{(\mu - \alpha)^2}{4\lambda} \right\} = p^{\mathrm{BZ}}(\beta, 0) + \frac{\mu^2}{4\lambda} \tag{3.7}$$

i.e. $\widetilde{\alpha}(\beta, \mu) = 0$. Therefore, by the Griffiths lemma and (3.1), (3.7) we get (3.4). □

Theorem 3.2. *Let* $\varepsilon_0 < 0$. *Then we have*

$$\rho_0(\theta, \mu) \equiv \lim_\Lambda \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_\Lambda} (\beta, \mu) = \left\{ \begin{array}{c} 0, \ \textit{for}\ (\theta, \mu) \in \widetilde{Q} \setminus \widetilde{D}_{\varepsilon_0} \\ \left(\frac{\widetilde{\alpha}(\beta,\mu) - \varepsilon_0}{g_0} \right), \ \textit{for}\ (\theta, \mu) \in \widetilde{D}_{\varepsilon_0} \end{array} \right\} \tag{3.8}$$

with $\widetilde{\alpha}(\beta, \mu)$ *defined by equation (3.5), see figure 2. Here domain* $\widetilde{D}_{\varepsilon_0}$ *is defined by*

$$\widetilde{D}_{\varepsilon_0} = \{(\theta, \mu) \in \widetilde{Q} : \varepsilon_0 < \widetilde{\alpha}(\beta, \mu)\} = \{(\theta, \mu) \in \widetilde{Q} : \widetilde{\mu}_0(\theta) < \mu\} \tag{3.9}$$

see figures 2 and 4, where we denote by $\widetilde{\mu}_0(\theta)$ *a unique solution of the equation*

$$\widetilde{\alpha}(\beta, \mu) = \varepsilon_0. \tag{3.10}$$

Proof. Since $\{p_\Lambda(\beta, \mu)\}_\Lambda$ are convex functions of $\varepsilon_0 \in \mathbb{R}^1$, then by

$$\left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_\Lambda} (\beta, \mu) = -\partial_{\varepsilon_0} p_\Lambda(\beta, \mu) \tag{3.11}$$

and by the Griffiths lemma (see [25, 26] or appendix B) we obtain that

$$\lim_\Lambda \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_\Lambda} (\beta, \mu) = -\partial_{\varepsilon_0} p(\beta, \mu). \tag{3.12}$$

For $\mu \leqslant \mu_c^{\mathrm{BZ}}(\theta) = 2\lambda\rho_c^{\mathrm{BZ}}(\theta)$ there is a unique $\widetilde{\alpha}(\beta, \mu) \leqslant 0$ defined by (3.5) which verifies (3.3), whereas for $\mu > \mu_c^{\mathrm{BZ}}(\theta)$ according to (3.7) we obtain $\widetilde{\alpha}(\beta, \mu) = 0$. Note that by (1.8) for $\mu \leqslant \varepsilon_0$ we have

$$p^{\mathrm{BZ}}(\beta, \mu) = p^{\mathrm{P}}(\beta, \mu).$$

Therefore, by (3.9), (3.12) one gets from (3.5) and (3.7) that

$$\lim_{\Lambda}\left\langle\frac{a_0^*a_0}{V}\right\rangle_{H_\Lambda}(\beta,\mu)=\left\{\begin{array}{c}0,\ \text{for}\ \widetilde{\alpha}(\beta,\mu)\leqslant\varepsilon_0<0\\ \left(\frac{\widetilde{\alpha}(\beta,\mu)-\varepsilon_0}{g_0}\right),\ \text{for}\ \varepsilon_0\leqslant\widetilde{\alpha}(\beta,\mu)\end{array}\right\}$$

i.e. (3.8). □

Hence by theorem 3.2, the domain $\widetilde{D}_{\varepsilon_0}$ (3.9) can be described as

$$\widetilde{D}_{\varepsilon_0}=\left\{(\theta,\mu)\in\widetilde{Q}:\rho_0(\theta,\mu)\equiv\lim_{\Lambda}\left\langle\frac{a_0^*a_0}{V}\right\rangle_{H_\Lambda}>0\right\}. \tag{3.13}$$

Note that in contrast to D_{ε_0}, see (1.9), (1.10), the domain $\widetilde{D}_{\varepsilon_0}$ has a temperature-dependent boundary and extends to positive μ, see figure 4. This macroscopic occupation of the mode $k=0$ (3.8) is a *non-conventional* Bose condensation which occurs in model (1.18) due to the 'attraction' term $\varepsilon_0 a_0^* a_0$, i.e. when $\varepsilon_0<0$ (see appendix A). It is *similar* to the *first* stage of condensation manifested by the model H_Λ^{BZ} (1.6) with $g_k(V)=g/2$), although in the latter case it is only possible for $\mu\leqslant 0$, see [13]. In particular, we once again have a saturation of the condensate density in the mode $k=0$:

$$\sup_{\mu\in\mathbb{R}^1}\rho_0(\theta,\mu)=\rho_0(\theta,\mu\geqslant\mu_c^{\mathrm{BZ}}(\theta))=-\frac{\varepsilon_0}{g_0} \tag{3.14}$$

see (1.15) and figure 3. Note that for any μ

$$\lim_{\beta\to0^+}\widetilde{\alpha}(\beta,\mu)=-\infty.$$

Thus, in contrast to model (1.6) (with $g_k(V)=g/2$), the non-conventional condensation in model (1.18) depends on the temperature. There is $\widetilde{\theta}_0(\mu)$ (the solution of the equation $\widetilde{\alpha}(\theta^{-1},\mu)=\varepsilon_0$, (3.10)) such that

$$\rho_0(\theta,\mu)=\frac{\widetilde{\alpha}(\beta,\mu)-\varepsilon_0}{g_0}>0 \tag{3.15}$$

for $\theta<\widetilde{\theta}_0(\mu)$ and

$$\rho_0(\theta,\mu)=0 \tag{3.16}$$

for $\theta\geqslant\widetilde{\theta}_0(\mu)$. This is another way to describe the phase diagram of model (1.18): $\widetilde{\theta}_0(\mu)$ is simply the inverse function of $\widetilde{\mu}_0(\theta)$, see figure 4.

Similar to (1.6), in model (1.18) for $d>2$ we encounter, for large total particle densities, *another kind* of condensation: a *conventional non-extensive* Bose–Einstein condensation in the vicinity of $k=0$ (see appendix A). In order to control this condensation we introduce an auxiliary Hamiltonian

$$H_{\Lambda,\gamma}\equiv H_\Lambda-\gamma\sum_{\{k\in\Lambda^*:\|k\|\geqslant\delta\}}a_k^*a_k \tag{3.17}$$

for a fixed $\delta>0$, and we set

$$p_\Lambda(\beta,\mu,\gamma)\equiv\frac{1}{\beta V}\ln\mathrm{Tr}_{\mathcal{F}_\Lambda}\,\mathrm{e}^{-\beta H_{\Lambda,\gamma}(\mu)}. \tag{3.18}$$

Remark 3.3. Let $\gamma<\varepsilon_\delta\equiv\varepsilon_{\|k\|=\delta}$. Then the system with Hamiltonian $H_{\Lambda,\gamma}$ has the same properties as the model H_Λ modulo the free-particle spectrum transformation:

$$\varepsilon_k\to\varepsilon_{k,\gamma}\equiv\varepsilon_k-\gamma\cdot\chi_{[\delta,+\infty)}(\|k\|) \tag{3.19}$$

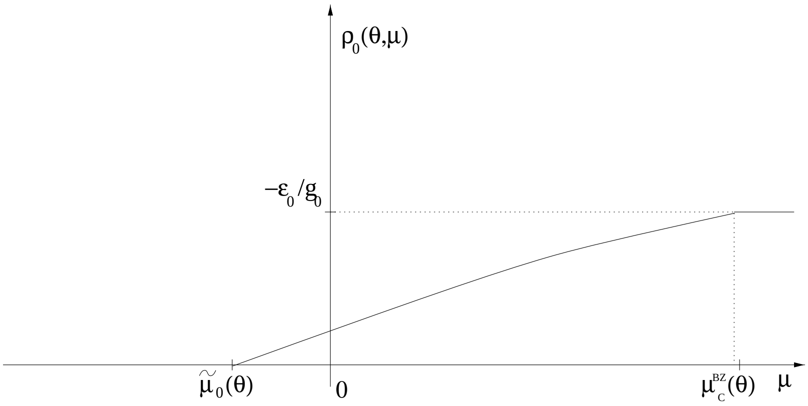

Figure 3. Non-conventional condensate density $\rho_0(\theta, \mu)$ as a function of the chemical potential μ and the temperature θ for the model H_Λ.

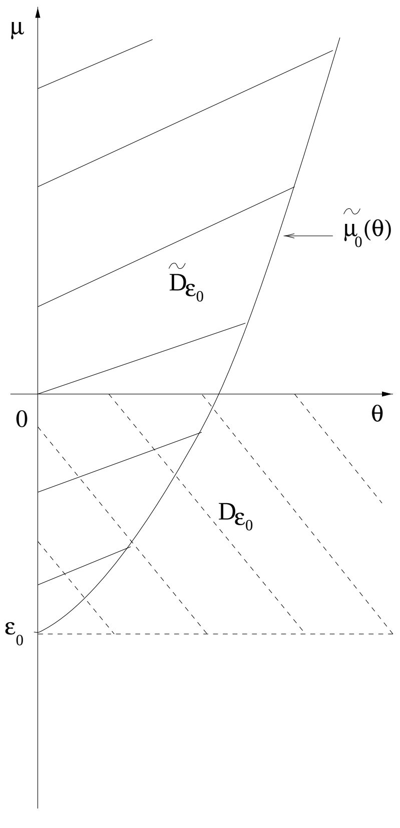

Figure 4. Domains D_{ε_0} and $\widetilde{D}_{\varepsilon_0}$ corresponding to the existence of a non-conventional condensation for the models H_Λ^{BZ} and H_Λ respectively.

where $\chi_A(x)$ is the characteristic function of domain A. In particular, the results of theorems 2.1 and 2.4 remain unchanged. For $(\theta, \mu) \in \widetilde{Q}$ and $\gamma < \varepsilon_\delta$ we have

$$p(\beta, \mu, \gamma) \equiv \lim_\Lambda p_\Lambda(\beta, \mu, \gamma) = \inf_{\alpha \leqslant 0} \left\{ p^{\mathrm{BZ}}(\beta, \alpha, \gamma) + \frac{(\mu - \alpha)^2}{4\lambda} \right\} \tag{3.20}$$

where $p^{\mathrm{BZ}}(\beta, \mu, \gamma)$ is the pressure (1.8) but with the free-particle spectrum (3.19):

$$\begin{aligned} p^{\mathrm{BZ}}(\beta, \mu, \gamma) &= p^{\mathrm{P}}(\beta, \mu, \gamma) - \inf_{\rho_0 \geqslant 0} \left[(\varepsilon_0 - \mu)\rho_0 + \frac{g_0 \rho_0^2}{2} \right] \\ &= \frac{1}{\beta(2\pi)^d} \int_{k \in \mathbb{R}^d} \ln[(1 - \mathrm{e}^{-\beta(\varepsilon_{k,\gamma} - \mu)})^{-1}]\, \mathrm{d}^d k - \inf_{\rho_0 \geqslant 0} \left[(\varepsilon_0 - \mu)\rho_0 + \frac{g_0 \rho_0^2}{2} \right]. \end{aligned} \tag{3.21}$$

Theorem 3.4. *For any $(\theta, \mu) \in \widetilde{Q}$ we have*

$$\lim_\Lambda \left\langle \frac{a_k^* a_k}{V} \right\rangle_{H_\Lambda} = 0 \qquad k \in \Lambda^* \setminus \{0\} \tag{3.22}$$

i.e., there is no macroscopic occupation of modes $k \neq 0$, whereas for $\mu > \mu_c^{\mathrm{BZ}}(\theta) = 2\lambda \rho_c^{\mathrm{BZ}}(\theta)$ the model H_Λ (1.18) manifests a generalized (non-extensive) Bose–Einstein condensation for those modes:

$$\lim_{\delta \to 0^+} \lim_\Lambda \frac{1}{V} \sum_{\{k \in \Lambda^* : 0 < \|k\| < \delta\}} \langle N_k \rangle_{H_\Lambda} = \rho(\beta, \mu) - \rho_c^{\mathrm{BZ}}(\theta) = \frac{1}{2\lambda}(\mu - \mu_c^{\mathrm{BZ}}(\theta)) > 0. \tag{3.23}$$

Here $\rho(\beta, \mu)$ is defined by (3.4). If $\varepsilon_0 < 0$, then this condensation coexists with the non-conventional condensation in the mode $k = 0$ (see theorem 3.2).

Proof. Let $g > 0$ and $\Delta g > 0$ be such that $g - \Delta g > 0$. Then by the Bogoliubov inequality (see e.g. [27]), one gets

$$0 \leqslant \frac{\Delta g}{2V^2} \sum_{k \in \Lambda^* \setminus \{0\}} \langle N_k^2 \rangle_{H_\Lambda} \leqslant p_\Lambda \left[H_\Lambda - \frac{\Delta g}{2V} \sum_{k \in \Lambda^* \setminus \{0\}} N_k^2 \right] - p_\Lambda[H_\Lambda]. \tag{3.24}$$

Note that by theorems 2.1 and 2.4 the thermodynamic pressure limits for two models, (1.18) with parameters $g > 0$ and $g - \Delta g > 0$, coincide with (2.13), i.e. one has

$$\lim_\Lambda \left\{ p_\Lambda \left[H_\Lambda - \frac{\Delta g}{2V} \sum_{k \in \Lambda^* \setminus \{0\}} N_k^2 \right] - p_\Lambda[H_\Lambda] \right\} = 0. \tag{3.25}$$

Since for any $k \in \Lambda^* \setminus \{0\}$ we have the estimate

$$0 \leqslant \left(\frac{\langle N_k \rangle_{H_\Lambda}}{V} \right)^2 \leqslant \frac{\langle N_k^2 \rangle_{H_\Lambda}}{V^2} \leqslant \frac{1}{V^2} \sum_{k \in \Lambda^* \setminus \{0\}} \langle N_k^2 \rangle_{H_\Lambda}$$

its combination with (3.24) and (3.25) gives (3.22).

Let $\delta > 0$, then we have

$$\frac{1}{V} \sum_{\{k \in \Lambda^* : 0 < \|k\| < \delta\}} \langle N_k \rangle_{H_\Lambda} = \rho_\Lambda(\beta, \mu) - \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_\Lambda} - \frac{1}{V} \sum_{\{k \in \Lambda^* : \|k\| \geqslant \delta\}} \langle N_k \rangle_{H_\Lambda}. \tag{3.26}$$

Now we can follow the same line of reasoning as in the proofs of theorems 3.1 and 3.2: we have the set $\{p_\Lambda(\beta, \mu, \gamma)\}_\Lambda$ of convex functions of $\gamma \in (-\infty, \varepsilon_\delta]$ with

$$\frac{1}{V} \sum_{\{k \in \Lambda^* : \|k\| \geqslant \delta\}} \langle N_k \rangle_{H_{\Lambda,\gamma}} = \partial_\gamma p_\Lambda(\beta, \mu, \gamma)$$

which by the Griffiths lemma and (3.20), (3.21) implies for $\gamma = 0$ that

$$\lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^* : \|k\| \geqslant \delta\}} \langle N_k \rangle_{H_\Lambda} = \partial_\gamma p(\beta, \mu, \gamma = 0). \tag{3.27}$$

Then by definitions (3.19), (3.20) and theorem 3.1 (see (3.2) and (3.4)), together with the explicit formula (1.13) we get for $\mu < \mu_c^{\mathrm{BZ}}(\theta)$ that

$$\partial_\gamma p(\beta, \mu, \gamma = 0) = \frac{1}{(2\pi)^d} \int_{\|k\| \geqslant \delta} \frac{\mathrm{d}^d k}{\mathrm{e}^{\beta(\varepsilon_k - \widetilde{\alpha}(\beta,\mu))} - 1} \tag{3.28}$$

and

$$\partial_\gamma p(\beta, \mu, \gamma = 0) = \frac{1}{(2\pi)^d} \int_{\|k\| \geqslant \delta} \frac{\mathrm{d}^d k}{\mathrm{e}^{\beta\varepsilon_k} - 1} \tag{3.29}$$

for $\mu \geqslant \mu_c^{\mathrm{BZ}}(\theta)$. Now, by virtue of (3.4), (3.14) and definition (1.14) we obtain (3.23) from (3.26), (3.27) and (3.29) by first taking the thermodynamic limit and then the limit $\delta \to 0^+$. □

4. Conclusion

We have presented a new exactly soluble model (1.18) which is inspired by the MSV model [12] and our previously presented model [13]. Due to an 'attractive'-type interaction in the mode $k = 0$ it belongs to the family of models which manifest two kinds of condensation: the *non-conventional* one in the mode $k = 0$ and *conventional* (generalized) Bose–Einstein condensation in modes $k \neq 0$. These condensations coexist for large total particle densities $\rho > \rho_c^{\mathrm{BZ}}(\theta)$, or $\mu \geqslant \mu_c^{\mathrm{BZ}}(\theta) = 2\lambda\rho_c^{\mathrm{BZ}}(\theta)$. This model demonstrates the richness of the notion of Bose condensation. It also gives a better understanding of the difference between *non-conventional* and *conventional* condensations. First, in spite of superstability of the model, which implies

$$\sup_{\mu \in \mathbb{R}^1} \rho(\beta, \mu) = +\infty$$

the conventional condensation is due to a mechanism of saturation. Since, after saturation of the non-conventional condensation, the kinetic-energy density attains its maximal value at the critical density $\rho_c^{\mathrm{BZ}}(\theta)$, the further growth of the total energy density for $\rho > \rho_c^{\mathrm{BZ}}(\theta)$ is caused by a macroscopic amount of particles with almost zero momenta.

The second important feature of model (1.18) (similar to [12, 13] and in contrast to [14]) is that the repulsion between bosons with $k \neq 0$ is strong enough to produce a generalized type III (i.e. non-extensive) Bose–Einstein condensation. Note that in the Bogoliubov weakly imperfect Bose gas [14, 15], the Bose–Einstein condensation is of type I.

The models of [12] and [13], together with the present one give explicit examples when the non-extensive Bose–Einstein condensation is produced by the interaction *between particles* and not by the geometry or by external field as in [7, 8] or in [6, 9, 10]. Note that the influence of different kinds of diagonal perturbations on Bose condensation has been a subject of careful analysis, see e.g. [28] and references therein. As, there, the authors characterize condensate by occupation measures (instead of occupation numbers), there is no way to establish the type of generalized condensation corresponding to a singular part of this measure. It is also an open problem whether the weight of this singular atomic measure is due to conventional or non-conventional condensation or to a combination of both.

Note added in proof. If one puts $g_0 = g = 0$, then our model (1.18) coincides with the model considered in chapter 4 of the paper [30]. According to our classification this model gives an example when condensation occurs in *one* stage: for $\mu > \mu_0^*(\theta) = 2\lambda p^P(\beta, \varepsilon_0) + \varepsilon_0$ one has only a *non-conventional* Bose condensation $p_0^*(\beta, \mu)$ which is not saturated because $g_0 = 0$, cf (3.14). A similar behaviour is obtained when $g_0 = \lambda = g = 0$, see [24], or when $\varepsilon_0 = g_0 = 0$ and $g = -\lambda < 0$ (the Huang–Yang–Luttinger), see [23, 28].

We thank Joe Pulè for attracting our attention to the paper [30] and for helpful comments.

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Appendix A. Classification of Bose condensations

A.1. The van den Berg–Lewis–Pulè's classification (condensations of type I, II and III)

For the reader's convenience we recall the nomenclature appertaining to (generalized) Bose–Einstein condensations according to [3, 7, 8]:

- the condensation is called the type I when a finite number of single-particle levels are macroscopically occupied;
- it is of type II when an infinite number of the levels are macroscopically occupied;
- it is called the type III, or the *non-extensive* condensation, when none of the levels are macroscopically occupied whereas one has

$$\lim_{\delta\to 0^+}\lim_{\Lambda}\frac{1}{V}\sum_{\{k\in\Lambda^*,0<\|k\|\leqslant\delta\}}\langle N_k\rangle = \rho - \rho_c(\theta).$$

An example of these different condensations is given in [7]. This paper demonstrates that three types of Bose–Einstein condensation can be realized in the case of the PBG in an *anisotropic* rectangular box $\Lambda \subset \mathbb{R}^3$ of volume $V = |\Lambda| = L_x \cdot L_y \cdot L_z$ and with Dirichlet boundary conditions. Let $L_x = V^{\alpha_x}$, $L_y = V^{\alpha_y}$, $L_z = V^{\alpha_z}$ for $\alpha_x + \alpha_y + \alpha_z = 1$ and $\alpha_x \leqslant \alpha_y \leqslant \alpha_z$. If $\alpha_z < \frac{1}{2}$, then for sufficiently large density ρ, we have the Bose–Einstein condensation of type I in the fundamental mode $k = (\frac{2\pi}{L_x}, \frac{2\pi}{L_y}, \frac{2\pi}{L_z})$. For $\alpha_z = \frac{1}{2}$ one gets a condensation of type II characterized by a macroscopic occupation of infinite package of modes $k = (\frac{2\pi}{L_x}, \frac{2\pi}{L_y}, \frac{2\pi n}{L_z})$, $n \in \mathbb{N}$, whereas for $\alpha_z > \frac{1}{2}$ we obtain a condensation of type III. In [6,9] it was shown that the type III condensation can be caused in the PBG by a weak external potential or (see [8,10]) by a specific choice of the boundary conditions and geometry. Another example of *non-extensive* condensation is given in [12, 13] for bosons in an *isotropic* box Λ, with *repulsive interactions* which spread out the *conventional* Bose–Einstein condensation of type I into Bose–Einstein condensation of type III.

A.2. Non-conventional versus conventional Bose condensation

Here we classify Bose condensations by their mechanisms of formation. In most papers (see [6–10, 12]), the condensation is due to *saturation* of the total particle density, originally discovered by Einstein [1] in the Bose gas without interaction (PBG). We call this *conventional* Bose–Einstein condensation [2].

The existence of condensation induced by *interaction* has been pointed out in some recent papers [13–15, 29]; it may also be the case for Huang–Yang–Luttinger or full diagonal models [28], since they contain attractive interactions. In particular, this is the case of the Bogoliubov weakly imperfect Bose gas [14]. We call this *non-conventional* Bose condensation.

(i) As has been shown in this paper (see also [13]), the non-conventional condensation does not exclude the appearance of the Bose–Einstein condensation when the total density of particles grows and exceeds some saturation limit $\rho_c^{\mathrm{BZ}}(\theta)$.

(ii) To appreciate the notion of non-conventional condensation let us remark that in models (1.6) and (1.18) for $d = 1, 2$, there only exists one kind of condensation, namely the non-conventional.

Since known Bose systems manifesting condensation are far from perfect, the concept of condensation induced by interaction is rather natural.

Remark A.1. A non-conventional Bose condensation can always be characterized by its type. Therefore, formally one obtains six kinds of condensation: the non-conventional versus the conventional of types I, II or III.

Appendix B. The Griffiths lemma [25, 26]

Lemma B.1. *Let* $\{f_n(x)\}_{n\geqslant 1}$ *be a sequence of convex functions on a compact* $I \subset \mathbb{R}$*. If there exists a pointwise limit*

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad x \in I \tag{B.1}$$

then

$$\begin{aligned} &\lim_{n\to\infty} \inf \partial_x f_n(x-0) \geqslant \partial_x f(x-0) \\ &\lim_{n\to\infty} \sup \partial_x f_n(x+0) \leqslant \partial_x f(x+0). \end{aligned} \tag{B.2}$$

Proof. By convexity one has

$$\begin{aligned} &\partial_x f_n(x+0) \leqslant \frac{1}{l}[f_n(x+l) - f_n(x)] \\ &\partial_x f_n(x-0) \geqslant \frac{1}{l}[f_n(x) - f_n(x-l)] \end{aligned} \tag{B.3}$$

for $l > 0$. Then taking the limit $n \to \infty$ in (B.3), by (B.1) we obtain

$$\begin{aligned} &\lim_{n\to\infty} \sup \partial_x f_n(x+0) \leqslant \frac{1}{l}[f(x+l) - f(x)] \\ &\lim_{n\to\infty} \inf \partial_x f_n(x-0) \geqslant \frac{1}{l}[f(x) - f(x-l)]. \end{aligned} \tag{B.4}$$

Now taking the limit $l \to +0$, in (B.4), one gets (B.2). □

Remark B.2. In particular, if $x_0 \in I$ is such that $\partial_x f_n(x_0 - 0) = \partial_x f_n(x_0 + 0)$ and $\partial_x f(x_0 - 0) = \partial_x f(x_0 + 0)$, then

$$\lim_{n\to\infty} \partial_x f_n(x_0) = \partial_x f(x_0).$$

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